Abstract Title Page

Title: How useful are fraction bars for understanding fraction equivalence and addition? A difficulty factors assessment with 5th, 6th, and 7th graders.

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Background / Context:

One strategy to promote deep understanding of math concepts has been to use multiple representations, particularly ones that connect to students' prior knowledge and aid sense-making. However, while these representations often seem intuitive for adults, there is not enough data on which representations make sense to students at what age. Singapore textbooks and the NCTM standards, for example, advocate using concrete visual representations in mathematics as a bridge to more formal, abstract thinking (NCTM, 2013; Leinwand & Ginsburg, 2007). Others caution that conceptual diagrams may not match student's mental models, and therefore may be difficult for students to interpret (Rittle-Johnson & Koedinger, 2001). Further, diagrams may harm performance among lower-ability students who may not have the competencies to understand them (Booth & Koedinger, 2011). To design effective instruction with diagrams, we must first understand what students need to know to make use of those diagrams.

This study investigates one specific type of diagram: fraction bars. To represent fraction *n/d* with a fraction bar, a unit-sized rectangle is divided into *d* equal pieces, with *n* pieces colored in. These types of diagrams are common in math curricula, including Everyday Math, Investigations (TERC) and Singapore math. Computer-based, interactive fraction bars in are also popular, for example NCTM's addition and equivalence applet (Suh, Moyer, & Heo, 2005). Our work with interactive fraction bars in a fraction addition tutoring system found that even though students learned from the tutor, they did not interpret the fraction bars correctly (Stampfer & Koedinger, 2012). The participants, all 5^{th} graders, often said that they were done solving a problem even though the fraction bar showing their inputted sum was not the same size as a fraction bar showing the true sum (Sample screenshot in Figure 1). That error occurred about 1-2 times per student per problem, leading us to investigate the cognitive mechanisms required for processing these representations. Our prior analyses with the 5^{th} grade data show that fraction bar scaffolds are helpful for determining fraction equivalence, but less so for evaluating a solved fraction addition problem (Stampfer & Koedinger, 2013). This study investigates how fraction bar utility develops as students advance from 5^{th} to 7^{th} grade.

Purpose / Objective / Research Question / Focus of Study:

How helpful are fraction bars for middle-school students evaluating fraction addition equations and fraction equivalence? How does fraction-bar understanding progress through middle school? To answer these questions, we compared three fraction bar formats and a numbers-only control condition among 5^{th} through 7^{th} graders. Since prior work indicated that 5^{th} graders understood all fraction bar formats for equivalence but not for addition, we wanted to see when students could successfully apply their fraction bar interpretation skills to the context of addition.

Setting:

We conducted the study in a local public school. Students completed the study materials independently during normal school hours in their usual classroom. The study materials were administered either by the first author or by the students' classroom teacher. All classrooms scheduled the study within a four-week period.

Population / Participants / Subjects:

All of the 5th through 7th grade classrooms at the school participated in the study (155 fifth graders, 145 sixth graders, and 153 seventh graders). Classrooms in each grade were tracked by the school and were described by teachers as inclusion, regular, and honors (Figure 2).

Intervention / Program / Practice:

Test items asked students to determine if two fractions were equivalent and to evaluate fraction addition equations. Fraction equivalence items presented two fractions and students indicated if the first fraction was bigger than, equivalent to, or smaller than the second fraction. Fraction addition items presented an equation with two different-denominator addends and a proposed sum. Students indicated if the equation was true or false.

Using a theoretical cognitive task analysis, we identified three likely skills needed to understand the fraction bar representations for fraction addition and equivalence: 1) equal areas represent equal amounts; 2) the rectangular bars represent the symbolic fractions written above or below them; and 3) if two shaded areas are equal, the fractions they represent are equal.

We developed test items intended to measure the difficulty of each skill (Figures 3-6). The **Pictures-Only** format (Figure 3) assesses if students know that the shaded rectangles use area to represent quantity, such that two rectangles with equal-sized shaded areas represent equal quantities. **Pictures-and-Numbers** items (Figure 4) include fraction symbols with the fraction bars, to test if students can understand the fraction bars as representations of fractions. **Half-Pictures-and-Numbers** items (Figure 5) also include both fraction bars and fraction symbols, but only presents the fraction bars as the hint at the top of the problem. This determines if students can find the relationship between the two fraction bars, map that relationship to the symbolic fractions represented, and then select the relationship that the symbolic fractions have to each other. **Numbers-Only** (Figure 6) provides a baseline for how well students can solve these problems without fraction bars.

For each of the four scaffold types for the addition equations, students were given one true equation and one false equation. The sums in the false equations followed the common misconception of adding both numerators and both denominators. For each of the four scaffold types for the equivalence items, students were given one pair of equivalent fractions and one pair of non-equivalent fractions. This paper discusses the experiment implemented with the 16 items described above. These items were part of a 30-item paper assessment that students were given 20 minutes to complete.

Research Design:

This study was designed as a difficulty factors assessment (cf., Koedinger, Alibali, & Nathan, 2008). By comparing performance on problems with different scaffold types, we can determine the difficulty of concepts or skills that are needed to solve one problem but not the other. For example, problems with the Pictures-Only scaffold test if students understand that areas represent amounts. Problems with the Pictures-and-Numbers scaffold also test understanding of amounts represented by areas, and additionally test the comprehensibility that those amounts are fractions. Comparing performance on these two items reveals the difficulty imposed by the fraction symbols.

The test forms were designed to avoid confounding: item scaffolds were counterbalanced with the specific numbers in the problems; item order was determined randomly; and half of the tests were given with the order reversed. Questions were scored as 1 if correct and 0 otherwise.

Data Collection and Analysis:

We collected students' answers and scores on all test items. We ran separate ANOVAs for each grade on the item scores: 3 (class tracking level: inclusion, regular, honors) x 4 (scaffold type: pictures only, pictures and numbers, half pictures and numbers, numbers only) x 2 (item: equivalence or addition) with repeated measures for the scaffold type and item, and with class tracking level as a between-subjects factor. Additionally, we ran two ANOVAs for all participants, one on addition item scores and one on equivalence item scores: 3 (grade: 5th, 6th, 7th) x 4 (scaffold type: pictures, pictures and numbers, half pictures and numbers, numbers only) with repeated measures for the scaffold type and grade as a between-subjects factor. We also did post-hoc Tukey tests to compare the grades to each other. For all repeated measures ANOVAs, we used the Huynh-Feldt correction for tests of within-subject effects since sphericity could not be assumed. We also ran separate ANOVAs on the addition item scores for each grade, with student as a random factor and scaffold as a fixed factor, and used post-hoc Tukey tests to compare the scaffold types to each other.

Findings / Results:

We ran separate repeated measures ANOVAs for each grade. They showed that item and scaffold were significant (both $p<.0005$). Additionally, there was a significant item by scaffold interaction for grade 5, a significant item by tracking level interaction for grade 6, and a significant scaffold by tracking level interaction for grades 6 and 7. For all grades, tests of between-subject effects showed that class tracking level was significant ($p<.0005$). These results indicate: 1) the equivalence items were easier than the addition items; 2) scaffold type affected difficulty for both question types; 3) for 5th graders, scaffold type affected difficulty differently for addition and equivalence items; 4) for 6th graders, the relative difficulty of the addition and equivalence problems depended on class tracking level; 5) for 6th and 7th graders, the relative difficulty of the different scaffold types depended on class tracking level; and 6) for all grades, students in higher tracks had higher overall scores. Figure 7 plots the mean for each question type by grade and scaffold, and figure 8 provides those means in a table.

The ANOVA on addition scores showed that scaffold was significant, and there was a significant scaffold by grade interaction (both $p<.0005$). Post-hoc Tukey tests show that the grades are significantly different from each other (all $p<.03$). These results indicate: 1) for the addition items, difficulty depended on scaffold type; 2) the relative difficulty of different scaffold types depended on grade; and 3) performance improved with grade level. When the ANOVA was repeated without the 5th grade, scaffold was again significant at the same level, but there was no scaffold by grade interaction. This indicates that the relative difficulty of different scaffold types was the same for 6th and 7th grade.

We ran separate ANOVAs for each grade on addition scores. Post-hoc Tukey tests from these analysis show different groupings for each grade. For 5th graders, performance was significantly different with each scaffold type. For 6th and 7th graders, numbers-only differed significantly from the other scaffolds, but the three scaffold types with pictures were not all significantly different from each other.

The ANOVA on equivalence scores showed that scaffold was significant ($p<.0005$), but there was not a significant scaffold by grade interaction. Post-hoc Tukey tests show that 5th and 7th grade performance were significantly different ($p<.0005$), but 6th grade performance was not significantly different from either. These results indicate: 1) for the equivalence items, difficulty depended on scaffold type; 2) the relative difficulty of different scaffold types did not depended on grade; and 3) students improved between 5th and 7th grade. Within each grade, performance with the three scaffold types that included pictures was almost the same, and performance with the numbers-only scaffold was significantly lower.

Conclusions:

This study investigated students' ability to make use of fraction bars, and how that ability develops from 5th to 7th grade. Since all participants attended the same school, our conclusions are still preliminary. Additionally, the study design has several limitations: the false addition equations all followed the same misconception of adding the numerators and denominators, and older students may have been taught specifically to avoid that error but might still fall for others. Also, the data does not fully untangle the skills of diagram interpretation from skills with fractions. Since both appear to improve with grade, it's difficult to tell if they are independent. Still, there are interesting differences in the pattern of performance across scaffold types.

We hypothesized that students must understand three aspects of fraction bars for that representation to be helpful: 1) equal areas represent equal amounts; 2) the rectangular bars represent the symbolic fractions written above or below them; and 3) if two shaded areas are equal, the fractions they represent are equal. 5th grade students appear to have all three competencies when deciding if two fractions are equivalent: they perform equally well with all scaffold types that include pictures, and the lower performance with the numbers-only control shows that they were not just calculating with the numbers. This pattern holds for 6th and 7th graders on equivalence items, with scores increasing steadily at each grade level. However, the pattern does not hold for fraction addition.

While the 5th graders performed equally well with all fraction bar scaffolds for equivalence, addition tested their diagram interpretation skills. Seeing the bars with fraction symbols decreased performance, and having to map the relationship between the bars to the relationship between the fractions decreased performance again. As students get older, they improve on all scaffold types and the difference in performance between scaffold types decreases, but does not disappear. The 5th grade students had substantial difficulty with the Pictures-and-Numbers scaffold (only 64% correct), perhaps explaining their confusion with the fraction bars in the tutor. 6th graders were significantly better with the two scaffold types that included fraction bars and numbers, suggesting the tutor's diagram-based feedback is better suited to their level. Overall, this study indicates that competencies for interpreting fraction bars are sensitive to context and develop slowly through middle school. This study also suggests that diagram-interpretation skills in general are context-sensitive. Based on these results we recommend giving 5th graders explicit instruction for understanding fraction bars, or delaying the use of interactive fraction bar feedback to 6th grade. Further, students who use diagrams successfully in one context may require additional support to transfer those skills to a new domain, even when the domains are closely related and the diagrams being used are similar.

Appendix A. References

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Appendix B. Tables and Figures

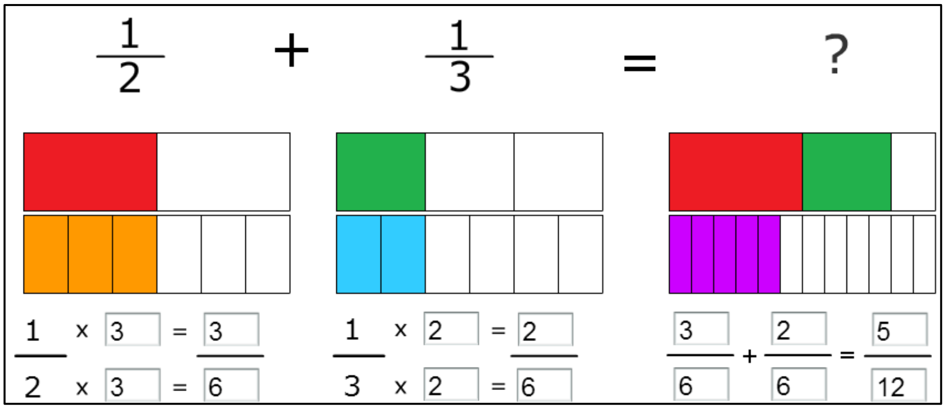

Figure 1: Fraction Addition Tutor. Top row of fractions and fraction bars are given, second row reflects students' inputs, typed in the boxes at the bottom. Text hints appear below when requested. Students often indicated they were done solving their addition problems even when their purple sum did not match the true sum (represented by the multicolored fraction bar).

	Inclusion	Regular	Honors	**Total**
5th Grade	37	61	57	**155**
6th Grade	34	58	53	**145**
7th Grade	35	63	55	**153**

Figure 2. Number of study participants by grade and school tracking level

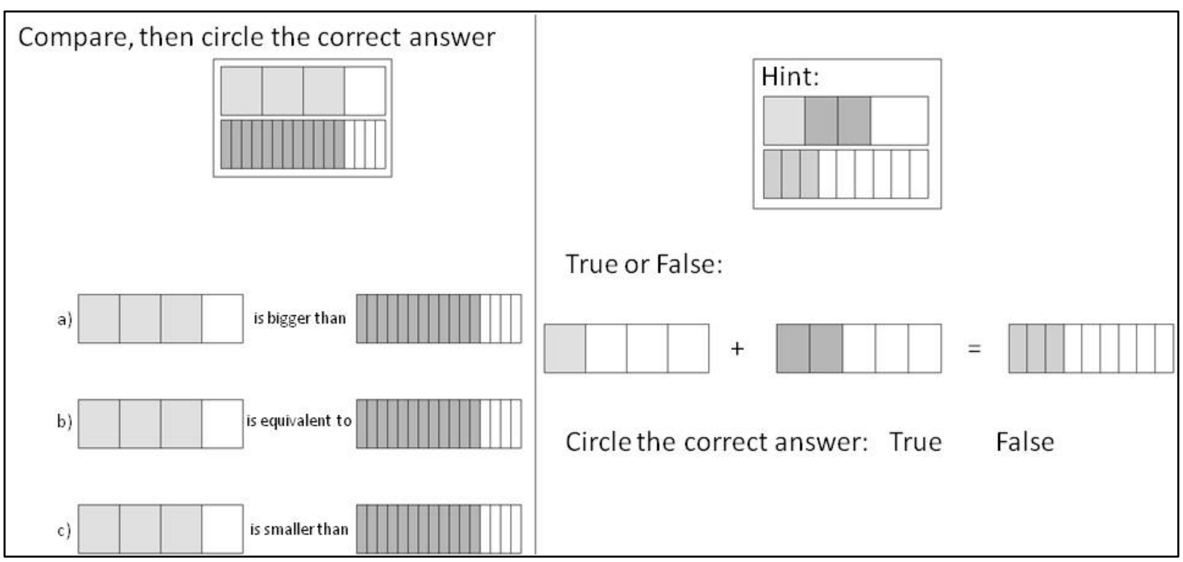

Figure 3. **Pictures-Only** format for equivalence and addition, testing if students understand that area represents quantity.

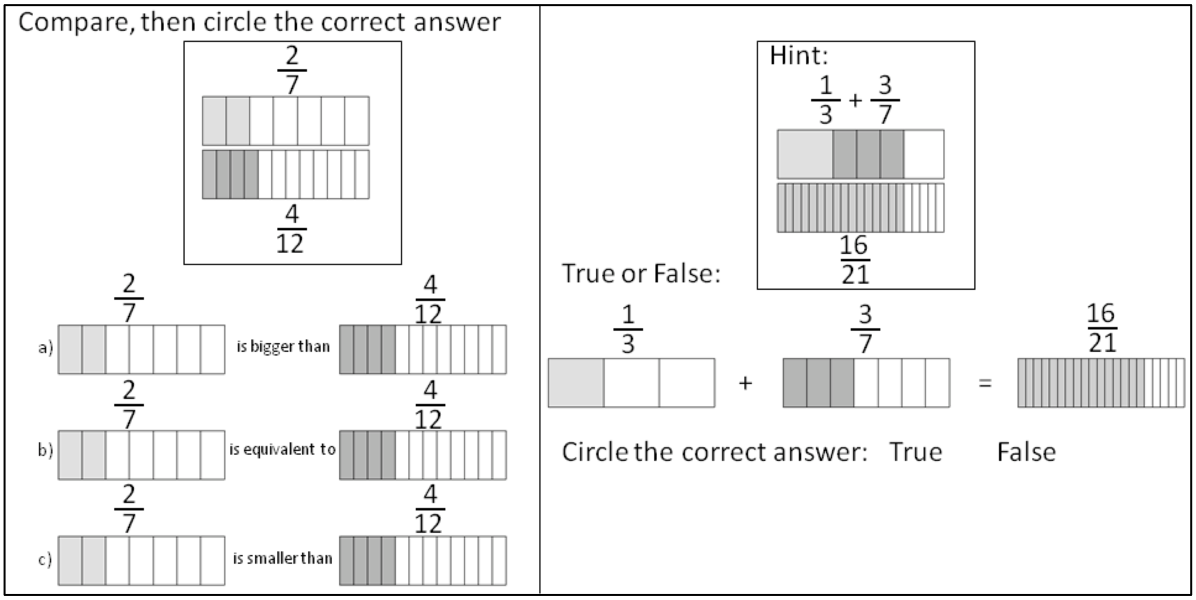

Figure 4. **Pictures-and-Numbers** format for equivalence and addition, testing if the images are comprehensible as fractions.

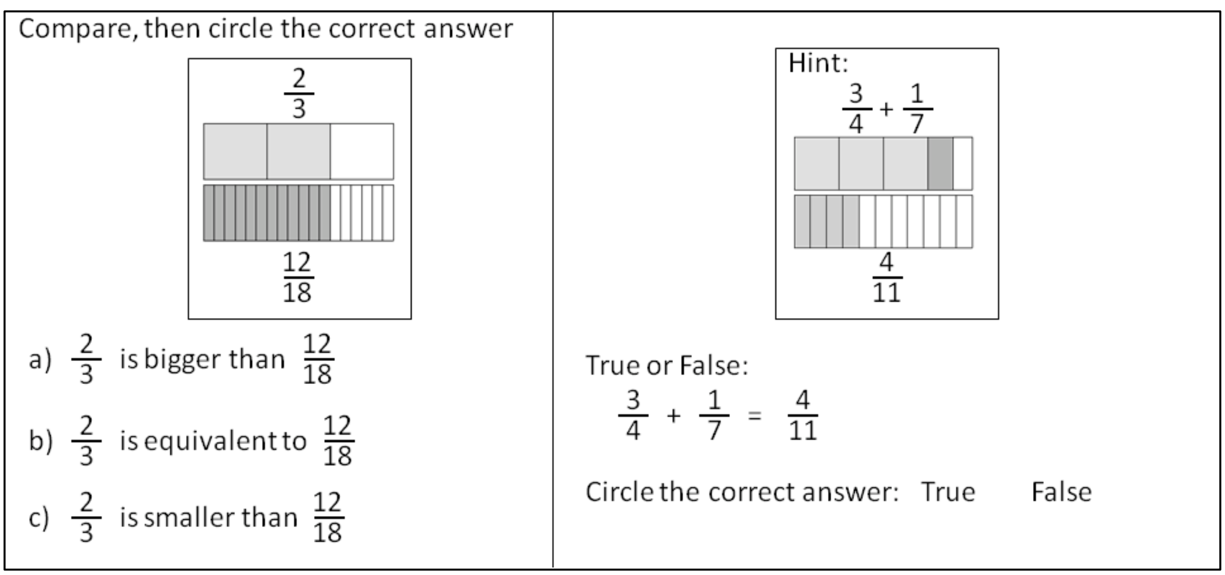

Figure 5. **Half-Pictures-and-Numbers** format for equivalence and addition, testing if students can map the relationship between the images to the relationship between the numbers.

Compare, then circle the correct answer	True or False:
a) $\frac{1}{3}$ is bigger than $\frac{8}{19}$	$\frac{2}{11} + \frac{1}{2} = \frac{15}{22}$
b) $\frac{1}{3}$ is equivalent to $\frac{8}{19}$	Circle the correct answer:
c) $\frac{1}{3}$ is smaller than $\frac{8}{19}$	True False

Figure 6. **Numbers-Only** control. Testing how well can students evaluate solved problems.

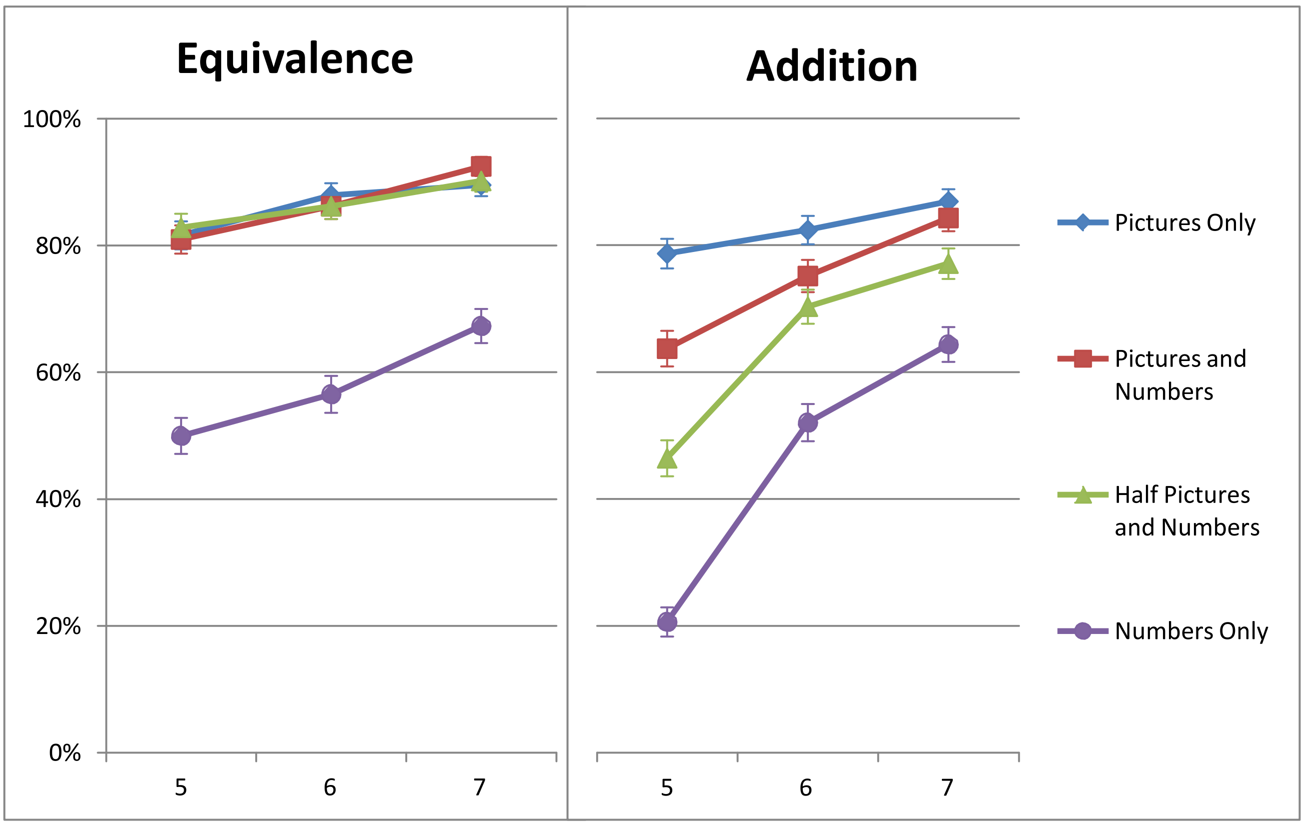

Figure 7. Mean equivalence and addition scores by grade and scaffold type, with standard error bars. Some lines or points overlap because the means are so close (see Figure 8).

	Equivalence			Addition		
	5th	**6th**	**7th**	**5th**	**6th**	**7th**
Pictures Only	81.6%	87.9%	89.5%	78.7%	82.4%	86.9%
Pictures and Numbers	81.0%	86.2%	92.5%	63.7%	75.2%	84.3%
Half Pictures and Numbers	82.8%	86.2%	90.2%	46.5%	70.3%	77.1%
Numbers Only	50.0%	56.6%	67.3%	20.6%	52.1%	64.4%

Figure 8. Table of means for equivalence and addition scores by grade and scaffold type

Appendix C. Acknowledgements
We thank our participants, their teachers, and Jo Bodner. This work was supported in part by the Pittsburgh Science of Learning Center through NSF award SBE-0836012, and by a Graduate Training Grant awarded to Carnegie Mellon University by the Department of Education (# R305B090023).